# Inclusion/Exclusion

Section 8.5+8.6

The hat problem: Suppose everyone one in a classroom has a hat, and they put their hat in a box when they enter, and then grab a random hat when they leave. What is the probability that everybody gets their own hat. Answer is $\frac{1}{n!}$

What about the probability that *someone* will not get their own hat: $1 - \frac{1}{n!}$

What about the probability that someone will get their own hat, or that no one? Need inclusion/exclusion. The probability that no one will get their own hat is 37.8% pretty much no matter how many people.

**Remember:**

$|A \cup B| = |A| + |B| - |A \cap B|$

and

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

## Formula

$$\begin{gathered}|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\ + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n)| \\ - |A_1 \cap A_2 \cap A_3 \cap A_4| - \ldots - |A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n| \\ + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| + \ldots + |A_{n-4} \cap A_{n-3} \cap A_{n-2} \cap A_{n-1} \cap A_n| \\ \ldots\ldots(-1)^{n+1}\ |A_1 \cap A_2 \cap \ldots \cap A_n|\end{gathered}$$

$$= \sum_{i=1}^{n} |A_i| - \sum_{i \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \ldots + (-1)^{k+1} \sum_{i_1 < i_2 < \ldots < i_k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}| + \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$$

Proof: Focus on any $x \in A_1 \cup A_2 \cup \ldots \cup A_n$ and see how many times it is counted. Let $k$ be the number of $i$ such that $x \in A_i$

$$-\binom{k}{0} + \binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \binom{k}{5} - \ldots + (-1)^{k+1}\binom{k}{k} + 1 = 1$$

(The reason we write +1 is because $-\binom{k}{0} = -1$)

Ignoring the +1, the rest is a row in Pascals triangle, and we now the alternating sum (+, then -, then +) is 0, so we know the whole thing will be 1.

Consider a set `A` with `N` elements. Properties $P_1, P_2, P_3, \ldots, P_n$ which means that some elements have the property and some doesn't. $N(P_3 P_5 P_6 P_9) =$ number of elements with each of the properties $P_3 P_5 P_6 P_9$

$N(P_3{}' P_5{}' P_6{}' P_9{}') =$ the number of elements with none of the properties $P_3 P_5 P_6 P_9$

**Inclusion/Exclusion:**

$$N(P_1{}' P_2{}' \ldots P_n{}') = N - \sum_{i=1}^{n} N(P_i) + \sum_{i_1 < i_2} N\left(P_{i_1} P_{i_2}\right) \ldots + (-1)^n N(P_1 P_2 \ldots P_n)$$

**Proof:** Put $A_i =$ the elements with property $P_i$

`m` balls in `n` boxes. We want to count the number of ways to put `m` balls into `n` boxes such that no box is empty. It follows $m \geq n$.

$A =$ all functions of $S \to B = \{1, 2, 3, ..., n\}$ $A_i =$ all functions such that `i` is not in the image.

$|A \setminus \cup_{i=1}^{n} A_i| = |A| - | \cup_{i=1}^{n} A_i| = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - ...(-1)^n \binom{n}{n}(n-n)^m$

# Permutations of an n-set {1,2,3,4}

Normally a permutation of a set is the elements put on a row (like here $2, 3, 1, 4$ and $3, 1, 4, 2$).

You can think of permutations as a bijection (different elements should be mapped to different elements, and every element needs to be mapped to an element) from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, 4\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

Notice the 4, mapped unto itself, it's called a fixed point. The $4'$s would also be a fixed point here:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

But then its a different set at the top.

## Derangement

A bijection ($\pi$) of a set `S` unto itself with no fixed points.

That is $\pi(i) \neq i$

The hat problem: to find the chances of noone getting their own hat, just use derangements.

## Counting permutations

$A$ is the set of all permutations of $\{1, 2, ..., n\}$

$A_i =$ the set of permutations $\pi$ such that $\pi(i) = i$

$$D_n = |A \setminus (\cup_{i=1}^{n}) = |A| - \sum_{i=1}^{n} + \sum_{i_1 < i_2} |A_{i_i} \cap A_{i_2}| - ...(-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$$

$$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + ...(-1)^k \binom{n}{k}(n-k)! \pm ... \; |A_1 \cap A_2 \cap ... \cap A_n|$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= k! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...(-1)^n \frac{1}{n!}\right]$$

$D_n$ is the permutations

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$
$$1 = 0.99999999\dots$$
$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$\exp(1) = e = 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$$
$$\exp(-1) = e^{-}1 = \frac{1}{e} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$$

$\exp(-1)$ is what we had before, so

$$D_n = k!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots(-1)^n \frac{1}{n!}\right] = n! \cdot \frac{1}{e}$$
$$= n!\frac{1}{2.7\dots} = 0.378n!$$

## Counting primes

The number of primes $\leq 100$. The same as saying $100 - \text{not primes}$

Put

$$A = \{1, 2, \dots, 100\}$$
$$A_2 = \text{The numbers divisible by } 2$$
$$A_3 = \text{The numbers divisible by } 3$$
$$A_5 = \text{The numbers divisible by } 5$$
$$A_7 = \text{The numbers divisible by } 7$$

Number of primes

$$= 100 - |A_2 \cup A_3 \cup A_5 \cup A_7| - 1 + 4 - |A_2| - |A_3| - |A_5| - |A_7| + |A_2 \cap A_3| + \dots$$
$$= 103 - 50 - 33 - 20 - 14 + 16 + 14 + 6 + \dots = 25$$

($-1$ because 1 is not a prime, and $+4$ because else we're excluding $2, 3, 5, 7$)