Funktioner 1

01002

Matematik 1b

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Funktioner

For at noget skal være en graf, må én input-værdi kun give én værdi.

Vektorfunktioner af flere variabler

Definition (3)

Funtioner af typen $\underline{f} : A \to \mathbb{R}^k$ har $A = \operatorname{dom}(f) \in \mathbb{R}^n$ kaldes vektorfunktioner af *n* variable.

$A = \operatorname{dom}(f)$ - Definitionsmængden (domæne)
$\mathbb{R}^k = \operatorname{codom}(f)$ - Dispositionsmængden (co-domænen)
$\operatorname{im}(f) = \operatorname{Vm}(f) = \left\{ \underline{f}(x) | \underline{x} \in \operatorname{dom}(f) \right\}$ - Værdimængden/billedrummet (image/range). Alt hvad funktionen kan ramme

$f = \underline{x} \mapsto f(\underline{x})$

Input: $\underline{x} \in \mathbb{R}^n$ - Søjlevektor i $\mathbb{R}^n$

Output: $\underline{f(x)} \in \mathbb{R}^k$ - Søjlevektor i $\mathbb{R}^k$

Example (1.3.1)

Lad $\boldsymbol{A} \in M_{k \times n}(\mathbb{R}) = \mathbb{R}^{k \times n}$

$$\begin{gathered} f : \mathbb{R}^n \to \mathbb{R}^k \\ f(\underline{x}) = A \cdot \underline{x} (f = \underline{x} \mapsto A\underline{x}) \end{gathered} \tag{1}$$

$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\begin{aligned} \boldsymbol{A}(\underline{x}) &= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_2 \end{aligned} \tag{2}$$

Ligning: $f(\underline{x}) = \underline{y} \quad , \underline{y} \in \operatorname{codom}(f) = \mathbb{R}^k$

Er den injektiv? Nej, $f\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Er den surjektiv? Nej, $\operatorname{im}(f) = \operatorname{span}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \operatorname{span}_{\mathbb{R}}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} | t \in \mathbb{R} \right\} = \operatorname{col}(\boldsymbol{A})$

Koordinatfunktioner

Example

For vektorfunktionen: $\boldsymbol{A}(\underline{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \end{bmatrix}$

$f_1(\underline{x}) = 1x_1 + 2x_2$

$$\begin{gathered} \underline{f}(\underline{x}) = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \\ \vdots \\ f_k(\underline{x}) \end{bmatrix} \\ f_i : \boldsymbol{A} \to \mathbb{R}, i = 1, ..., k \\ \boldsymbol{A} = \mathrm{dom}(f) \end{gathered} \tag{3}$$

Example (1.2.2)

$f : \boldsymbol{A} \to \mathbb{R}, f(x_1, x_2) = \sqrt{16 - x_1^2 - x_2^2}$

hvor $\boldsymbol{A} = \mathrm{dom}(f) \in \mathbb{R}^2$

Typisk opgave: Hvad er $\boldsymbol{A}, \mathrm{codom}(f), \mathrm{im}(f)$

Find størst mulig $\boldsymbol{A}$

$$\begin{gathered} 16 - x_1^2 - x_2^2 \geq 0 \\ 4^2 = 16 \geq x_1^2 + x_2^2 \end{gathered} \tag{4}$$

$\mathrm{codom}(f) = \mathbb{R}$

$$\begin{gathered} f(0,0) = \sqrt{16 - 0^2 - 0^2} = 4 \\ f(4,0) = \sqrt{16 - 4^2 - 0^2} = 0 \end{gathered} \tag{5}$$

Så $\mathrm{im}(f) = [0, 4]$

Example (1.3.2 (ReLu))

$$\begin{gathered} \mathrm{ReLu} : \mathbb{R} \to \mathbb{R} \\ \mathrm{ReLu}(x) = \max(0, x) = \begin{cases} x \ , \ x \geq 0 \\ 0 \ , \ x < 0 \end{cases} \end{gathered} \tag{6}$$

$$\begin{gathered} \mathrm{ReLu} : \mathbb{R}^n \to \mathbb{R}^n \\ \mathrm{ReLu}(\underline{x}) = \begin{bmatrix} \mathrm{ReLu}(x_1) \\ \mathrm{ReLu}(x_2) \\ \vdots \\ \mathrm{ReLu}(x_n) \end{bmatrix} \end{gathered} \tag{7}$$

Visualiseringer af funktioner (1.4 i bogen)

$\mathrm{Graf}(f) = \left\{ \left(\underline{x}, \underline{f}(\underline{x}) \right) \in \mathbb{R}^{n+k} \mid \underline{x} \in \mathrm{dom}\left(\underline{f} \right) \right\}$

Se Eks 1.2.2. Ville kræve $n + k \leq 3$

Niveaukurver

Niveaukurver for $c \in \mathrm{codom}\left(\underline{f} \right)$ - c: output

$\left\{ \underline{x} \in \mathrm{dom}(f) \mid \underline{f}(\underline{x}) = \underline{c} \right\}$

Neuralt netværk (Def 1.3.2)

1. lag: $\underline{x} \mapsto \sigma_1(\boldsymbol{A}_1 \underline{x} + \underline{b}_1) = \underline{z}_1$

$\boldsymbol{A}_1 \in \mathbb{R}^{50 \times 784}$

$$\begin{gathered} f_1 : \mathbb{R}^{784} \to \mathbb{R}^{50} \\ \sigma_1 : \mathbb{R}^{50} \to \mathbb{R}^{50} \\ \sigma_1 = \mathrm{ReLu} \end{gathered} \tag{8}$$

2. lag:

$$\begin{gathered} \underline{z}_1 \mapsto \sigma_2(\boldsymbol{A}_2 \underline{z}_1 + \underline{b}_2) = \underline{z}_2 \in \mathbb{R}^{10} \\ \boldsymbol{A}_2 \in \mathbb{R}^{10 \times 50}, \underline{b}_2 \in \mathbb{R}^{10 \times 1} \end{gathered} \tag{9}$$

$$\begin{gathered} \Phi = f_2 \circ f_1, \quad R^{894} \to \mathbb{R}^{10} \\ f \circ g(x) = f(g(x)) \\ \Phi(\underline{x}) = \sigma_2(\boldsymbol{A}_2 \sigma_1(\boldsymbol{A}_1 \underline{x} + \underline{b}_1) + \underline{b}_2) \end{gathered} \tag{10}$$

Kontinuitet

Definition (3.2.1)

En funktion $f : \mathrm{dom}(f) \to \mathbb{R}, \quad \mathrm{dom}(f) \in \mathbb{R}$ er kontinuert i $x_0 \in A = \mathrm{dom}(f)$ hvis $x \to x_0 \Rightarrow f(x) \to f(x_0)$ (ligning 3.1)

Ligning 3.1 betyder at for ethvert $\varepsilon > 0$ findes der et $\delta > 0$ således at $|\, x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$$\forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \varepsilon \Rightarrow |f(x) - f(x_0)| < \delta \tag{11}$$

Funktionen f er kontinuert hvis den er kontinuert i alle punkter i $\mathrm{dom}(f)$

Example

Lad $a, b \in \mathbb{R}$

$$f(x) = ax + b, \quad f : \mathbb{R} \to \mathbb{R} \tag{12}$$

Påstanden: f er kontinuert

Solution:

$$\begin{aligned} |f(x) - f(x_0)| &= |\ ax + b(ax_0 + b)| \\ &= |ax - ax_0| = |a|\ |x - x_0| \end{aligned} \tag{13}$$

Lad ε være givet. Vi vælger $\delta = \frac{\varepsilon}{|a|}$. Så

$$\begin{gathered} |x - x_0| < \delta = \frac{\varepsilon}{|a|} \\ \Rightarrow |f(x) - f(x_0)| = |a||x - x_0| \\ < |a|\ \frac{\varepsilon}{|a|} = \varepsilon \end{gathered} \tag{14}$$

Example

$$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = \begin{cases} 1 & x \Rightarrow 0 \\ 0 & x < 0 \end{cases} \tag{15}$$

h er diskontinuert i $x_0 = 0$. Lad $\varepsilon = \frac{1}{2}$.

Der findes intet $\delta > 0$:

$$\begin{gathered} |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{1}{2} \\ |x| < \delta \Rightarrow |f(x) - 1| < \frac{1}{2} \end{gathered} \tag{16}$$

Så

$$\begin{gathered} f\left(\frac{-\delta}{2}\right) = 0 \\ f\left(\frac{\delta}{2}\right)) = 1 \end{gathered} \tag{17}$$

$$\begin{gathered} |f\left(\frac{-\delta}{2}\right) - f(0)| \\ = |0 - 1| = 1 \end{gathered} \tag{18}$$

Så funktionen er ikke kontinuert.

Example

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{19}$$

Er f kontinuert for alle $x,y) \in \mathbb{R}^2$?

Solution:

Opstil y som graf af x, så $y = ax, \quad a \in \mathbb{R}$. Så fås

$$\begin{aligned} f(x,ax) &= \frac{x^2ax}{x^4+a^2x^2} \\ &= \frac{ax}{x^2+a^2} \\ &= \frac{ax}{x^2+a^2} \to \frac{0}{0+a^2} = 0, x \to 0 \end{aligned} \tag{20}$$

Hvis $y = ax^2, \quad a \in \mathbb{R}$

$$\begin{aligned} f(x,ax^2) &= \frac{x^2ax^2}{x^4+a^2x^4} \\ &= \frac{a}{1+a^2} \end{aligned} \tag{21}$$

Så funktionen er ikke kontinuert $(x,y) = (0,0)$