Induction

Examples

Prove $2^0+2^1+2^2+\dots+2^n=2^{n+1}-1$

Basis step: $P(0)$: $1=2^{0+1}-1$ which is true

Inductionstep: $P(k)\to P(k+1)$

assume $P(k)$ is true for some $k\in\mathbb{N}$,

$$\underbrace{1+2+\dots+2^k}_{=2^{k+1}-1}+2^{k+1}=2^{k+1}-1+2^{k+1}$$
$$=2\cdot 2^{k+1}-1$$
$$=2^{k+2}-1$$
$$=2^{(k+1)+1}-1$$

Now it is shown that its true

Harmonic numbers: $H_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$

Prove: $H_{2^n}\geq 1+\frac{n}{2}$ for all $n\in\mathbb{N}$

Basisstep: $H_{2^0}=H_1\geq 1+\frac{0}{n}$ is true

Inductionstep: Assume $H_{2^k}\geq 1+\frac{k}{2}$ and we want to show this implies $H_{2^{k+1}}\geq 1+\frac{k+1}{2}$

$$H_{2^{k+1}}=\underbrace{1+\frac{1}{2}+\dots+\frac{1}{2^k}}+\frac{1}{2^k+1}+\dots+\frac{1}{2^{k+1}}$$
$$=H_{2^k}+\frac{1}{2^k+1}+\dots+\frac{1}{2^{k+1}}$$
$$\geq 1+\frac{k}{2}+\underbrace{\frac{1}{2^k+1}+\dots+\frac{1}{2^{k+1}}}$$

Show that this is at least $\frac{1}{2}$ which would make the whole thing greater than $1+\frac{k+1}{2}$

Replace it with its smallest value $\frac{1}{2^k+1}$ is at least $\frac{1}{2^0+1}$ or $\frac{1}{2^{k+1}}$

$$\geq 1+\frac{k}{2}+\frac{1}{2^{k+1}}+\dots+\frac{1}{2^{k+1}}=1+\frac{k}{2}+2^k\left(\frac{1}{2^{k+1}}\right)$$
$$=1+\frac{k}{2}+\frac{1}{2}=1+\frac{k+1}{2}$$

Suppose $2n+1$ people are standing at distinct distances from each other

Prove that if each person throws a pie at the person nearest to them, then there's at least one survivor.

Suppose $n=1$. there will be two people whos distance between eachother is the smallest. they will throw at each other, so third person is fine. Basis step is fine

Inductive step: Assume $P(k)$ is true

consider $2(k+1)+1=2k+3$ people.

Let A and B be the closest pair, so they will throw at eachother.

Case 1: No one else throws a pie at A or B, then the remaining $2k+1$ people are the same as when A and B were there.

we can do it again and again, so there will always be a survivor by our induction hypothesis

Case 2: at least one person throws a pie at A or B. Then at least 3 pies are thrown at A and B, but this means there remains at most $2k+3-3=2k$ pies to be thrown at the other $2k+1$ people. So at least one will survive

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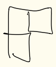
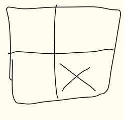
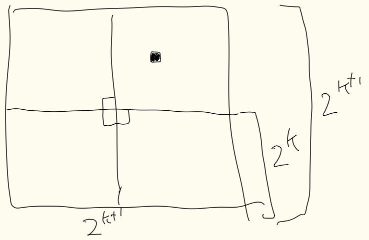
Show that every $2^n\times 2^n$ checkerboards with one square removed can be tiled by right triominoes 

Basis Step: $n=1$  this can be tiled.

Inductive Step: Assume we can tile it. consider a $2^{n+1}\times 2^{n+1}$ checkerboard with one square removed.



Strong induction

Sometimes there may be more than 1 basis step

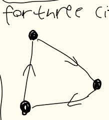
Basis step: prove $P(1)$ is true

Induction step: prove $[P(1) \wedge P(2) \wedge \dots \wedge P(k)] \to P(k+1)$

Example

A country with n cities, any two of those cities are connected by a one-way road. Show that there is a route that goes through every city

for three cities



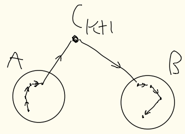
$P(3)$ is true

Inductive step: Assume it's true for any number up to k cities. We will show it's true for $k+1$ cities

consider $C_1, C_2, \dots, C_{k+1}$

$A = \{C_i : C_i \text{ has a road to } C_{k+1}\}$

$B = \{C_i : C_i \text{ has a road from } C_{k+1}\}$



For A and B there must be a road that goes around it

Recursively defined functions

Have **basis step:** specify $f(0)$ or more

Recursive step/rule: give a rule for determining the next value in the function.

Often we write f_n instead of $f(n)$

Prove that $f_n > \alpha^{n-2}$ where $\alpha = \frac{1+\sqrt{5}}{2}$ for $n \geq 3$

$$f = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x = 1 \\ f_{n-1} + f_{n-2} & \text{for } x \geq 2 \end{cases}$$

Basis step: $f_3 = 2 \quad \alpha^{3-2} = \alpha$

$f_4 = 3 \quad \alpha^{4-2} = \alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2$

$= \frac{1+5+2\sqrt{5}}{4} = \frac{6+2\sqrt{5}}{4}$

$= \frac{3+\sqrt{5}}{2} < \frac{3+3}{2} = 3$

Inductive step: Assume $f_j > \alpha^{j-2}$

$\forall\ 3 \leq j \leq k$. What to show $(k \geq 4)$

$f_{k+1} > \alpha^{(k+1)-2}$

$f_{k+1} = f_k + f_{k-1}$

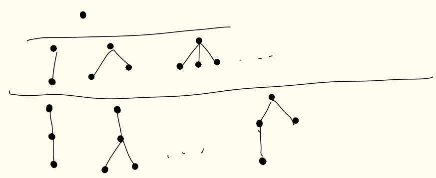
$> \alpha^{k-2} + \alpha^{k-3}$

$= \alpha^{k-3}(\alpha + 1)$

$= \alpha^{k-3}\alpha^2$

$= \alpha^{k-1} = \alpha^{(k+1)-2}$

Recursively defined sets and structures



Similar to functions

Basis step: Specify some initial elements

Recursive step: Rules for forming new elements in the set from those already known to be in the set

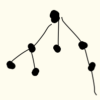
Example

Basis step: $3 \in S$

Recursive step: if $x, y \in S$ then $x+y \in S$

$S = \{3k \mid k \in \mathbb{Z}^+\}$

Rooted tree



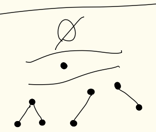
Recursive definition

Basis step: A single node is a rooted tree with root r

Recursive step: suppose $T_1, T_2, \dots, T_n$ are rooted trees with roots $r_1, r_2, \dots, r_n$

Then the structure, formed by starting with a root r (which is new) and adding an edge from r to each r_i, is a rooted tree.

Extended binary trees

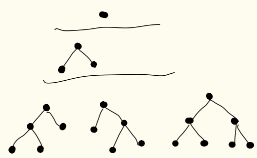


Every node has only 1 or 2 children

Basis step: The empty tree is an EBT

Recursive step: If T_1 and T_2 are EBTs, then there is an EBT $T_1 \cdot T_2$ consisting of a root r, together with edges connecting r with the roots of $T_1 + T_2$

Full binary tree



Basis step: a single node r is a FBT

Recursive step: same as for EBT

Height of FBT

Def height $h(T)$ of a FBT T

basis $h(\cdot) = 0$

recursive: $h(T_1 \cdot T_2) = 1 + \max\{h(T_1), h(T_2)\}$

Amount of nodes

$n(\cdot) = 1 \quad n(T_1 \cdot T_2) = 1 + n(T_1) + n(T_2)$

Then: if T is a FBT, then $n(T) \leq 2^{h(T)+1} - 1$

Basis step: $n(\cdot) = 1 \quad h(\cdot) = 0 \quad 1 \leq 2^{0+1} - 1$

Inductive step: Assume $n(T_1) \leq 2^{h(T_1)+1} - 1 \to n(T_2) \leq 2^{h(T_2)+1} - 1$

Let $T = T_1 \cdot T_2$ and $r = \max\{h(T_1), h(T_2)\}$

$n(T) = 1 + n(T_1) + n(T_2)$

$\leq 1 + 2^{h(T_1)+1} - 1 + 2^{h(T_2)+1}$

$\leq 2^{r+1} + 2^{r+1} - 1$

$= 2^{r+2} - 1 = 2^{(r+1)+1} - 1 = 2^{h(T)+1} - 1$

Structual induction

Basis step: show result holds for all elements specified in basis step of the defn.

Recursive step: show that if the result holds for each of the elements used to construct a new element, then it holds for the new element

Opgaver

5.1: 19, 51, 75, 65

5.2: 3, 25

5.3: 1, 5, 7, 13, 17, 31, 45

5.2 25

a) 3, 5, ...

b) $\mathbb{Z}^+$

c) 2, 4, 8, ...

d) $\mathbb{Z}^+$

5.3 5

a. No. N-2 could be negative

b. $1-n$. $f(n+1) = f(n) - 1 = (1-n) - 1 = 1-(n+1)$

c. $4-n$. $f(n) - 1 = 4-n-1 = 4-(n+1)$

d. $2^{(n-1)}$. $2f(n-1) = 2 \cdot 2 \cdot 2^{n-1-1} = 2^n$

e.

19

a: $P(2): 1 + \frac{1}{4} < 2 - \frac{1}{2}$

b: $\frac{5}{4} < \frac{6}{4}$ True

c: $1 + \frac{1}{4} + \frac{1}{9} + \dots \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$ for $k \geq 2$

d:

e: $1 + \frac{1}{4} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left[\frac{1}{k} - \frac{1}{(k+1)^2}\right] =$

$$= 2 - \left[\frac{k^2+2k+1-k}{k(k+1)^2}\right] = 2 - \left[\frac{k^2+k}{k(k+1)^2} - \frac{1}{k(k+1)^2}\right]$$

$$= 2 - \left[\frac{k(k+1)}{k(k+1)^2} - \frac{1}{k(k+1)^2}\right] = 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} < 2 - \frac{1}{k+1}$$

f. because induction

51

the $x-1 = y-1$ could be negative numbers

75

a) $\frac{1}{2} < 1$

$$\sum_{j=1}^{k+1} j/(j+1)! = \sum_{j=1}^{k} j/(j+1)! + \frac{k+1}{(k+2)!}$$