Lecture - November 20, 2025

01017
Discrete Mathematics

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Sections 9.5 and 9.6: Relations and Partial Orders

Introduction to Relations

Definition (Relation)

Consider a set S and a relation R on S. We write (S, R) to denote this structure.

For $a, b \in S$, we say "a is related to b" and write aRb if the ordered pair (a, b) is in the relation R.

In essence, a relation is just a set of ordered pairs: $R \subseteq S \times S$.

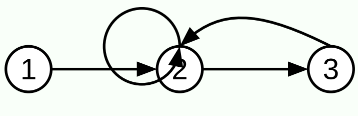
Example (Finite Relation)

Let $S = \{1, 2, 3\}$ and suppose R contains the pairs $(1, 2)$, $(2, 3)$, $(3, 2)$, and $(2, 2)$.

We can represent this as:

$$R = \{(1, 2), (2, 3), (3, 2), (2, 2)\} \tag{1}$$

This relation can be visualized as a directed graph:



Common Sets

Throughout this section, we use the following standard sets:

$$\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{Q}, \mathbb{Q}_+, \mathbb{R}, \mathbb{C} \tag{2}$$

We also work with power sets $\mathcal{P}(S)$, which denote the set of all subsets of S.

Common Relations

Examples of relations include:

$$=, \leq, <, \geq, >, \in, \subseteq, \subset, \supseteq, \supset, \equiv, | \tag{3}$$

Equivalence Relations

Definition (Equivalence Relation)

A relation R on a set S is an **equivalence relation** if and only if R satisfies the following three properties:

1. **Reflexive:** $\forall a \in S, aRa$
2. **Symmetric:** $\forall a, b \in S$, if aRb then bRa
3. **Transitive:** $\forall a, b, c \in S$, if aRb and bRc then aRc

Example (Classifying Relations)

Let's classify common relations by their properties:

Reflexive Relations:

$$=, \leq, \geq, \subseteq, \supseteq, \equiv \tag{4}$$

Symmetric Relations:

$$=, \equiv \tag{5}$$

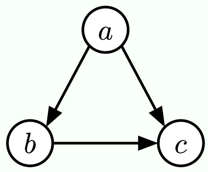
Transitive Relations:

$$=, \leq, \geq, \subseteq, \supseteq, \equiv, | \tag{6}$$

Note: The relation $|$ on $\mathbb{Z}_+$ is reflexive and transitive, but not symmetric (e.g., $2 \mid 4$ but $4 \nmid 2$).

Example (Transitivity Visualization)

If aRb and bRc, then by transitivity we must have aRc:



Equivalence Classes

Definition (Equivalence Class)

If $\sim$ is an equivalence relation on S, the **equivalence class** of an element $a \in S$ is denoted $[a]_\sim$ and defined as:

$$[a]_\sim = \{x \in S \mid x \sim a\} \tag{7}$$

This is the set of all elements related to a.

Example (Congruence Modulo 4)

Consider the equivalence relation of congruence modulo 4 on $\mathbb{Z}$:

$$\begin{aligned}
[0]_{\bmod 4} &= \{\cdots, -8, -4, 0, 4, 8, \cdots\} \\
[1]_{\bmod 4} &= \{\cdots, -7, -3, 1, 5, 9, \cdots\} \\
[2]_{\bmod 4} &= \{\cdots, -6, -2, 2, 6, 10, \cdots\} \\
[3]_{\bmod 4} &= \{\cdots, -5, -1, 3, 7, 11, \cdots\}
\end{aligned} \tag{8}$$

These four equivalence classes partition the integers.

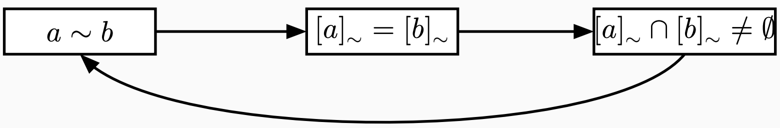
Theorem (Properties of Equivalence Classes)

Let $\sim$ be an equivalence relation on S. For any $a, b \in S$, the following are equivalent:

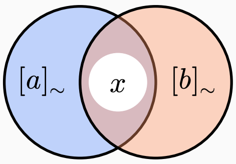
1. $a \sim b$
2. $[a]_\sim = [b]_\sim$
3. $[a]_\sim \cap [b]_\sim \neq \emptyset$

Proof:

We prove the cycle of implications: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.



(3) ⇒ (1): Suppose $[a]_\sim \cap [b]_\sim \neq \emptyset$. Then there exists some $x \in [a]_\sim \cap [b]_\sim$.



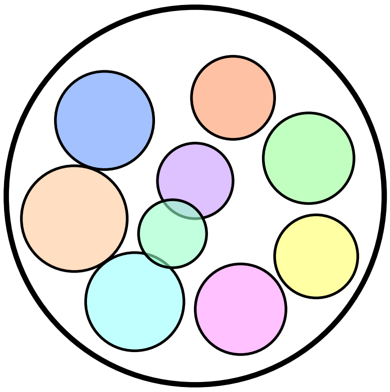
By definition, $x \sim a$ and $x \sim b$. By symmetry, $a \sim x$. By transitivity, $a \sim x$ and $x \sim b$ implies $a \sim b$. □

Theorem (Partition Property)

Let $\sim$ be an equivalence relation on a set S. The equivalence classes of $\sim$ form a partition of S.

That is:

1. Every element $a \in S$ belongs to exactly one equivalence class $[a]_\sim$.
2. The equivalence classes are pairwise disjoint: if $[a]_\sim \neq [b]_\sim$, then $[a]_\sim \cap [b]_\sim = \emptyset$.
3. The union of all equivalence classes equals S.



Partial Orders (Posets)

Definition (Partially Ordered Set (Poset))

A relation R on a set S is a **partial order** if R satisfies:

1. **Reflexive:** $\forall a \in S, aRa$
2. **Antisymmetric:** $\forall a, b \in S$, if aRb and bRa then $a = b$
3. **Transitive:** $\forall a, b, c \in S$, if aRb and bRc then aRc

We call (S, R) a **partially ordered set** or **poset**.

Note

The key difference between equivalence relations and partial orders is:
- Equivalence relations require **symmetry**
- Partial orders require **antisymmetry**

Example (Examples of Partial Orders)

The following are partial orders:

$$=, \leq, \geq, \subseteq, \supseteq, | \text{ (on } \mathbb{Z}_+) \tag{9}$$

For instance, $(\mathbb{Z}_+, |)$ is a poset because:
- $\forall a \in \mathbb{Z}_+$: $a \mid a$ (reflexive)
- If $a \mid b$ and $b \mid a$, then $a = b$ (antisymmetric)
- If $a \mid b$ and $b \mid c$, then $a \mid c$ (transitive)

Notation and Terminology

For a partial order R on S, we often use the notation $\preceq$ instead of R, and write $a \preceq b$ instead of aRb.

Note

Different authors may use different symbols such as $\leq$ or $\sqsubseteq$ for general partial orders. The symbol $\preceq$ is commonly used when the specific ordering is understood from context.

Definition (Comparable Elements)

Two elements $a, b \in S$ are **comparable** if either $a \preceq b$ or $b \preceq a$.

If a and b are not comparable, we say they are **incomparable**.

Definition (Total Order (Chain))

A partial order $(S, \preceq)$ is a **total order** or **chain** if every pair of elements is comparable.

That is, for all $a, b \in S$, either $a \preceq b$ or $b \preceq a$.

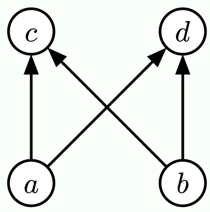
Definition (Well-Ordered Set)

A poset $(S, \preceq)$ is **well-ordered** if every non-empty subset of S has a smallest element (minimum).

Example (Incomparable Elements)

Consider the poset shown below:

Maximal elements



Minimal elements

In this poset:

- a and b are **incomparable** (neither $a \preceq b$ nor $b \preceq a$)
- c and d are **incomparable**
- Both a and b are **minimal elements**
- Both c and d are **maximal elements**

Minimal and Maximal Elements

Definition (Minimal and Maximal Elements)

Let $(S, \preceq)$ be a poset.

- An element $a \in S$ is **minimal** if there is no element $x \in S$ such that $x \preceq a$ and $x \neq a$.

- An element $a \in S$ is **maximal** if there is no element $x \in S$ such that $a \preceq x$ and $x \neq a$.

Definition (Minimum and Maximum Elements)

Let $(S, \preceq)$ be a poset.

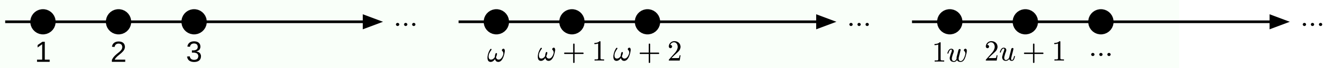
- An element $a \in S$ is the **minimum** (or **least element**) if $\forall x \in S, a \preceq x$.
- An element $a \in S$ is the **maximum** (or **greatest element**) if $\forall x \in S, x \preceq a$.

Note

Key differences:
- A poset can have multiple minimal/maximal elements
- A poset has at most one minimum/maximum element
- Every minimum is minimal, but not every minimal element is a minimum

Example (Ordinal Numbers)

Consider the ordinal numbers with their natural ordering:



This is a well-ordered set where every subset has a minimum element.

Theorem (Existence of Minimal Elements)

Every finite partially ordered set has at least one minimal element.

Proof:

Let $(S, \preceq)$ be a finite poset. Pick any element $x_1 \in S$.

- If x_1 is minimal, we are done.
- Otherwise, there exists $x_2 \preceq x_1$ with $x_2 \neq x_1$.
- If x_2 is minimal, we are done.
- Otherwise, continue this process.

Since S is finite, this process must terminate, yielding a minimal element.

Geometrically, we can think of placing elements on a vertical line where if $x \preceq y$, then x is below y:

The bottommost element in such a representation is minimal. □

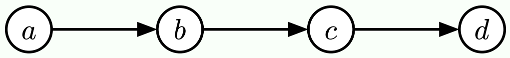
Topological Sorting

Definition (Topological Sort)

A **topological sort** of a finite poset $(S, \preceq)$ is a total ordering of the elements that is compatible with the partial order.

That is, we arrange elements in a sequence $a_1, a_2, \cdots, a_n$ such that if $a_i \preceq a_j$ in the partial order, then $i \leq j$ in the sequence.

Example (Topological Sort)



Topological sort

This represents a compatible total ordering $a \preceq b \preceq c \preceq d$.

Lexicographic Ordering

Definition (Lexicographic Order)

Given two partially ordered sets $\left(S_1, \underset{1}{\preceq}\right)$ and $\left(S_2, \underset{2}{\preceq}\right)$, we can define the **lexicographic ordering** $\preceq$ on $S_1 \times S_2$ as follows:

$(a_1, a_2) \preceq (b_1, b_2)$ if and only if either:

- $a_1 \underset{1}{\prec} b_1$, or
- $a_1 = b_1$ and $a_2 \underset{2}{\preceq} b_2$

Example (Alphabetical Ordering)

The lexicographic order generalizes alphabetical ordering. With letters ordered as $A < B < C < \cdots < Z$:

$$\text{ANNA} < \text{ANNI} < \text{JOHN} \tag{10}$$

This is because:

- “ANNA” vs “ANNI”: First three letters match, but N < N is false and the fourth letter A < I
- “ANNI” vs “JOHN”: First letter A < J

Hasse Diagrams

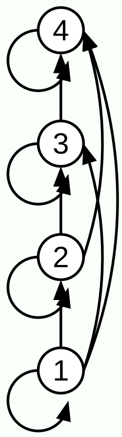
Definition (Hasse Diagram)

A **Hasse diagram** is a visual representation of a finite poset that simplifies the traditional directed graph by:

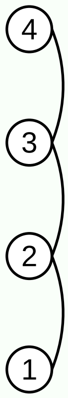
1. Removing all self-loops (reflexivity is implicit)
2. Removing all transitive edges (transitivity is implicit)
3. Removing all arrows (direction is upward by convention)
4. Positioning elements so that if $x \preceq y$, then x is below y

Example (Poset ${1,2,3,4}$ with $<=$)

First, the complete directed graph:



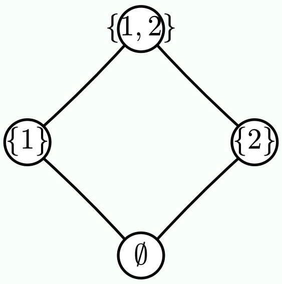
The corresponding Hasse diagram (with redundant information removed):



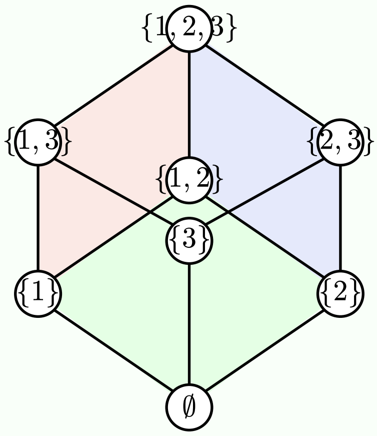
The Hasse diagram is much cleaner while preserving all ordering information.

Example (Power Set Hasse Diagrams)

Hasse diagram for $(\mathcal{P}(\{1,2\}), \subseteq)$:



Hasse diagram for $(\mathcal{P}(\{1,2,3\}), \subseteq)$:



Notice how the structure forms a cube, reflecting the Boolean lattice structure of power sets.

Exercises

Section 9.5

11

Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more

Solution:

For this we need to show R is reflexive, symmetric and transitive.

1. **Reflexive:** Every bit string in R of length three or more, agrees with itself in the first three bits. Hence $(x, x) \in R$ for all x.
2. **Symmetric:** For all a and B in R of length three of more that agrees with their first three bits, a agress with b in its first three digits, and also vice verca (because the first three digits are the same).

3. **Transistive:** Same as before, if a and b agree in their first three digits and so does b and c, then so does a and c because they must have the same first three digits.

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What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010
b) 1011
c) 11111
d) 01010101

Solution:

a) It would consist of all bit strings starting with 010. So like $0100001, 010110, 010111, \text{etc.}$
b) It would consist of all bit strings starting with 101.
c) It would consist of all bit strings starting with 111
d) Same as a) because here its first three digits are 010

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What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is:

a) 2?
b) 3?
c) 6?
d) −3?

Solution:

a) $\{..., -8, -3, 2, 7, 12, ...\}$
b) $\{..., -7, -2, 3, 8, 13, ...\}$
c) $\{..., -9, -4, 1, 6, 11\}$
d) Samme som a)?

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Which of these are partitions of the set $\mathbb{Z} \times \mathbb{Z}$ of ordered pairs of integers?

a) *the set of pairs (x, y), where x or y is odd; the set of pairs (x, y), where x is even; and the set of pairs (x, y), where y is even*
b) *the set of pairs (x, y), where both x and y are odd; the set of pairs (x, y), where exactly one of x and y is odd; and the set of pairs (x, y), where both x and y are even*
c) *the set of pairs (x, y), where x is positive; the set of pairs (x, y), where y is positive; and the set of pairs (x, y), where both x and y are negative*
d) *the set of pairs (x, y), where $3 \mid y$ and $3 \mid y$; the set of pairs (x, y), where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y), where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y), where $3 \nmid x$ and $3 \nmid y$*
e) *the set of pairs (x, y), where $x > 0$ and $y > 0$; the set of pairs (x, y), where $x > 0$ and $y \leq 0$; the set of pairs (x, y), where $x \leq 0$ and $y > 0$; and the set of pairs (x, y), where $x \leq 0$ and $y \leq 0$*

f) *the set of pairs (x, y), where $x \neq 0$and $y \neq 0$; the set of pairs (x, y), where $x = 0$ and $y \neq 0$; and the set of pairs (x, y), where $x \neq 0$ and $y \neq 0$*

Solution:

For them to be partitions, they need to uphold:
1. Every element $a \in S$ belongs to exactly one equivalence class $[a]_\sim$.
2. The equivalence classes are pairwise disjoint: if $[a]_\sim \neq [b]_\sim$, then $[a]_\sim \cap [b]_\sim = \emptyset$.
3. The union of all equivalence classes equals S.

a) Not this one as there is an intersection between the sets (when x is even, and y is odd. This set is in the first two)
b) This is a partition, as there is no overlap between them (you have three sets containing only even x and y, only odd x and y, and all the other cases where one of x and y is odd).
c) Not this one, as the set of pairs where x is positive can contain positive y, so there's overlap.
d) This is a partition, same reason as b).
e) This is a partition, same reason as b)

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Section 9.6

3

Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if
a) *a is taller than b?*
b) *a is not taller than b?*
c) *$a = b$ or a is an ancestor of b?*
d) *a and b have a common friend?*

Solution:

For this to be true it needs to follow:
1. **Reflexive:** $\forall a \in S, aRa$
2. **Antisymmetric:** $\forall a, b \in S$, if aRb and bRa then $a = b$
3. **Transitive:** $\forall a, b, c \in S$, if aRb and bRc then aRc

a) This isn't reflective as a is not taller than a
b) This is reflective, for the reason as in a). This wouldn't be antisymmetric as people can exist that are the exact same height.
c) This is reflective. This is also antisymmetric as the only way for aRb and bRa is if the first condition of $a = b$ is met. This would also be transitive as if a is an ancestor of b, who is an ancestor of c, then a is also an ancestor of c.
d) This is reflective as your friends are all common friends with yourself. This wouldn't be antisymmetric as when you have a common friend with someone, they also have that same common friend with you

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Find the lexicographic ordering of the bit strings $0, 01, 11, 001, 010, 011, 0001, \text{and } 0101$ *based on the ordering* $0 < 1$.

Solution:

$0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$

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Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

a) *Find the maximal elements.*
b) *Find the minimal elements.*
c) *Is there a greatest element?*
d) *Is there a least element?*
e) *Find all upper bounds of* $\{3, 5\}$.
f) *Find the least upper bound of* $\{3, 5\}$, *if it exists.*
g) *Find all lower bounds of* $\{15, 45\}$.
h) *Find the greatest lower bound of* $\{15, 45\}$, *if it exists.*

Solution:

The poset would include $\{(3, 9), (3, 15), (3, 24), (3, 45), (5, 15), (5, 45), (9, 45)\}$
a) 24 and 45. They can't divide any number.
b) 3 and 5. They arent divisible by any number.
c) No. The closest would be 45, but it isnt divisible by 24.
d) No. The closest would be 3, but it doesnt divide 5.
e) $(15, 45)$
f) 15
g) $(3, 5, 15)$
h) 15

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