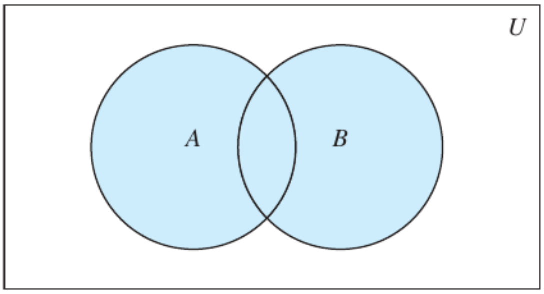
Union of 2 sets

Def: Let A and B be sets. The Union of A and B, denoted by $A \cup B$, is the set that contains every element in either A or B or both.

$$A \cup B = \{x | x \in A \vee x \in B\}$$

The order of the operation doesn't matter



$A \cup B$ is shaded.

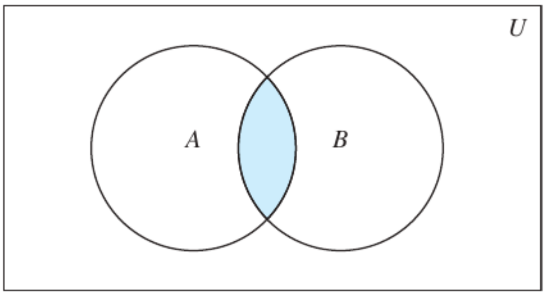
FIGURE 1 **Venn diagram of the union of A and B.**

Intersections

Def: Let A and B be sets. The intersection of A and B, denoted by $A \cap B$, is the set containing those elements, that are in both A and B.

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

The order of the operation doesn't matter



$A \cap B$ is shaded.

FIGURE 2 **Venn diagram of the intersection of A and B.**

Disjoint

Def. A and B are disjoint if $A \cap B = \emptyset$, aka they don't have any elements in common.

Size of elements in two sets

If A and B are finite sets, then:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

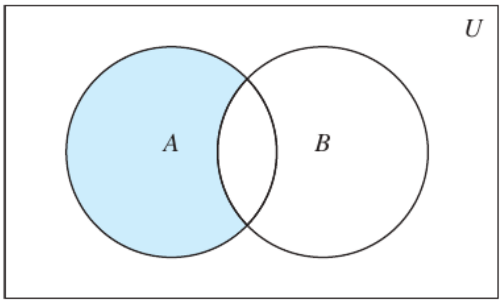
To get size of the elements in the sets, you need to add their size together, and then subtract all the elements they have in common.

Difference

Def: Let A and B be sets. The Difference of A and B, denoted $A - B$, is the set containing those elements of A that are not in B.

$$A - B = \{x | x \in A \land x \notin B\}$$

Sometimes also denoted $A \setminus B$



$A - B$ is shaded.

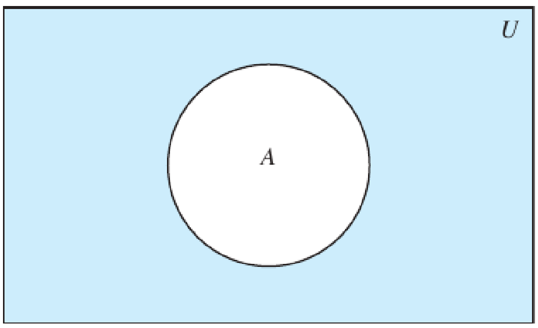
FIGURE 3 **Venn diagram for the difference of A and B.**

If $A - B = \emptyset$. Either $A = B$ or $A \subset B$ (A is a subset of B).

Complement

Everything that is not in the set. But you need a universal set (in the pictures, the rectangle is the universal set).

Def: Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the containing all the elements of U that are not in A. $\bar{A} = U - A$



$\bar{A}$ is shaded.

FIGURE 4 **Venn diagram for the complement of the set A.**

Example.

$$\begin{aligned} U = \mathbb{N}, \quad & A = \{1, 3, 5, 7, ...\} \\ & \bar{A} = \{0, 2, 4, 6, ...\} \\ U = \mathbb{Z}, \quad & \bar{A} = \{..., -2, -1, 0, 2, 4, ...\} \end{aligned}$$

Set identities

TABLE 1 Set Identities.	
Identity	***Name***
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Prove: $\overline{A \cup B} = \bar{A} \cup \bar{B}$

$$x \in \overline{A \cup B} \Leftrightarrow$$
$$x \notin A \cup B \Leftrightarrow$$
$$x \notin A \land x \notin B \Leftrightarrow$$
$$x \in \bar{A} \land x \in \bar{B} \Leftrightarrow$$
$$x \in \bar{A} \cup \bar{B}$$

If x is in the complement of the union of A and B, then it must not be in the union of A and B. In other words, it is not in either A or B. This can be written as x being in the complement of A and the complement of B. This can be then be written as x is in the union of the complement of A and the complement of B.

Prove: $\overline{A \cup (B \cap C)} = \left(\bar{C} \cup \bar{B}\right) \cap \bar{A}$

$$\overline{A \cup (B \cap C)} \Leftrightarrow$$
$$\bar{A} \cap \left(\overline{B \cup C}\right) \Leftrightarrow$$
$$\bar{A} \cap \left(\bar{B} \cup \bar{C}\right)$$

Unions and intersections of an arbitrary number of sets

$\cup_{i=1}^{\curvearrowleft} A_i = \{x | \exists\ i \in \{1, ..., b\} \text{ such that } x \in A_i\}$

$\cap_{i=1}^{\curvearrowleft} A_i = \{x | \forall\ i \in \{1, ..., n\} \text{ such that } x \in A_i\}$

$\cup_{i=1}^{\infty} A_i = \{x | \exists\ i \in \mathbb{Z}^+ \text{ such that } x \in A_i\}$

$\cap_{i=1}^{\infty} A_i = \{x | \forall\ i \in \mathbb{Z}^+ \text{ such that } x \in A_i\}$

Example: $A_i = \{1, 2, 3, ..., i\}, A_1 = \{1\}, A_2 = \{1, 2\}$

$\cup_{i=1}^{\infty} A_i = \mathbb{Z}^+, \cap_{i=1}^{\infty} A_i = \{1\}$

Functions

Def: Let A and B be nonempty sets. A function *f* from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the element of B that *f* assigns to *a*.

We write $f : A \to B$ to denote that *f* is a function from A to B.

$(a, f(a)) \in A \times B$

Def: Let $f : A \to B$

Domain of *f* is A

Codomain of *f* is B

If $f(a) = b$, then b is the **image** of *a* under *f*, and *a* is a **preimage** of *b*. For every *a* it matches a single *b*.

Range/Image of *f* the set of all images

Example: $A = \{a, b, c, d\},\ B = \{1, 2, 3, 4\},\ f : A \to B$

$f(a) = 1,\ f(b) = 2,\ f(c) = 3,\ f(d) = 4$

Real-valued function means that codomain $= \mathbb{R}$ (real number)

Integer-valued function means that codomain $= \mathbb{Z}$ (integers)

Sum

Def: let f_1 and f_2 be realvalued functions from A, then $f_1 + f_2$ and $f_1 f_2$ are functions from A to $\mathbb{R}$ defined as:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$f_1 f_2(x) = f_1(x) f_2(x)$$

Example: $f_1, f_2 : \mathbb{R} \to \mathbb{R}\ f_1(x) = x^2,\ f_2(x) = x - x^2$

$$(f_1 + f_2)(x) = x$$
$$f_1 f_2(x) = x^3 - x^4$$

Def: Let $f : A \to B$. IF $S \subseteq A$, then the image of S under f, denoted $f(S)$ is the set $\{b \in B | \exists a \in S \text{ such that } f(a) = b\}$

Example: $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$

$S = \{-1, 0, 1\}$

$f(S) = \{0, 1\}$

Injectivity and Surjectivity

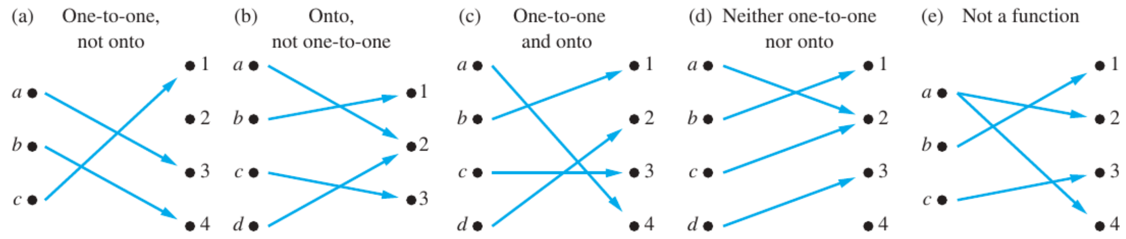


FIGURE 5 **Examples of different types of correspondences.**

Injectivity

Def: A function $f : A \to B$ is one-to-one/injective/an injection if $f(x) = f(y) \Rightarrow x = y, \forall x, y \in A$. Two elements in the domain, cannot match to the same element in the codomain.

$x \neq y \Rightarrow f(x) \Rightarrow f(y)$.

Example

$f : \mathbb{Z} \to \mathbb{Z}$ $f(x) = x^2$

This is not injective as for example both −1 and 1 give the same answer.

$f : \mathbb{N} \to \mathbb{Z}$ $f(x) = x^2$ would be injective as you now only have positive x.

Def: Let $f : A \to B$ such that $A, B \subseteq \mathbb{R}$. Then f is increasing if $x \leq y \Rightarrow f(x) \leq f(y)$ $\forall x, y \in A$.

It would be strictly increasing if $x < y \Rightarrow f(x) < f(y)$ $\forall x, y \in A$. And this would be the same for decreasing, only the opposite of course.

A strictly increasing function implies that the function is injective

Surjectivity

Def: A function $f : A \to B$ is surjective/onto/a surjection if $\forall b \in B, \exists a \in A$ such that $f(a) = b$. For every element in the codomain, there exists an element in the domain that matches to it.

Example: $f : \mathbb{Z} \to \mathbb{Z}$ $f(x) = x - 1$

Let $k \in \mathbb{Z}$, then $k + 1 \in \mathbb{Z}$ and $f(k + 1) = k + 1 - 1 = k$.

$f : \mathbb{N} \to \mathbb{N}$ $f(x) = 2x$ would not be surjective because the uneven numbers in the codomain would not have an element in the domain matched to them. $f : \mathbb{Z} \to \mathbb{Z}$ $f(x) = 2x$ would be surjective because for every real number, half that number would still be a real number, so every element in the codomain would have an element in the domain that matches it.

Bijection

Def: A function is a bijection/one-to-one correspondence if it is both injective and surjective

Suppose $f : A \to B$

To show that f is *injective*: show if $f(x) = f(y)$, then $x = y$ (for arbitrary $x, y \in A$)

To show that it's not *injective*: Find a particular $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$

To show that f is *surjective*: Consider arbitrary $b \in B$, and show $\exists a \in A$ such that $f(a) = b$

To show that it's not *surjective*: Find a particular $b \in B$ such that $\nexists a \in A$ with $f(a) = b$

Inverse

Def: Let $f : A \to B$ be a bijection. The inverse function of *f*, denoted f^{-1}, is the function from B to A that assigns to $b \in B$ the unique element $a \in A$ such that $f(a) = b$. Note: $f^{-1} \neq \frac{1}{f}$.

Composition

Let $f : A \to B,\ g : B \to C$

$$g \circ f : A \to C$$
$$f \circ g(a) = g(f(a))$$

Floor function

Largest integer that is less than or equal to x

Ceiling function

Smallest integer that is greater than or equal to x