01019 E24 Exam - Discrete Mathematics

01017

Discrete Mathematics

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DTU (Technical University of Denmark)

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Der anvendes en scoringsalgoritme, som er baseret på “One correct answer”

Dette betyder følgende:
- Der er altid præcist ét korrekt svar
- Studerende kan kun vælge ét svar per spørgsmål
- Hvert rigtigt svar giver 1 point. Så et spørgsmål, der består af f.eks. 3 del-spørgsmål, giver 3 points, hvis alle 3 er korrekte.
- Hvert forkert svar giver 0 point (der benyttes IKKE negative point)

The following approach to scoring responses is implemented and is based on “One correct answer”:
- There is always only one correct answer
- Students are only able to select one answer per question
- Every correct answer that you click on corresponds to 1 point, so a question with 3 parts is worth 3 points if you get every part correct.
- Every incorrect answer corresponds to 0 points (incorrect answers do not result in subtraction of points)

Question 1

If a, b, c, and d are positive integers such that $ab \mid cd$, then which of the following must be true?

Vælg en svarmulighed

- ○ $a|\text{lcm}(c, d)$ or $b|\text{lcm}(c, d)$
- ● If p is a prime that divides a, then $p|c$ or $p|d$
- ○ None of these
- ○ If b is prime, then $b|\gcd(c, d)$
- ○ $\gcd(a, b)|\ \gcd(c, d)$
- ○ $a|c$ or $a|d$ or $b|c$ or $b|d$

Solution:

Because if you instead of a, b, c, d put them factorized into primes. Then if ab divides cd they must share some common primes when factorized, and thus if p divides a so it is in the factorized ab, then it must also be in the factorized cd and thus $p \mid c$ or $p \mid d$

Question 2

For $n \geq 4$, the number of edges in the n-cube Q_n is

Vælg en svarmulighed

- ○ None of these
- ○ $2^n + 16$
- ● $n2^{n-1}$
- ○ 2^{n+1}
- ○ $n! + 8$

Solution:

It it probably not the 2nd or last one. 3rd one makes more intuitive sense than 4th one.

Question 3

For each of the following, determine whether it is surjective/injective, or not a well defined function. Recall that $\mathbb{N}$ is the set of natural numbers, in other words the set of nonnegative integers, and $\mathbb{Z}^+$ is the set of positive integers.

Vælg de rigtige svarmuligheder

	Not well defined	Well defined but neither	Surjective but not injective	Injective but not surjective	Both surjective and injective
$f : \mathbb{Z}^+ \to \mathbb{N}$ given by $f(x) = \lfloor \log_2(x) \rfloor$	○	○	●	○	○
$f : \mathbb{R} \to \mathbb{Z}^+$ given by $f(x) = \lfloor \lvert x \rvert \rfloor$	●	○	○	○	○
$f : \mathbb{N} \to \mathbb{Z}$ given by $f(x) = \begin{cases} \lceil x/2 \rceil & \text{if } x \text{ is even} \\ -\lceil x/2 \rceil & \text{if } x \text{ is odd} \end{cases}$	○	○	○	○	●
$f : \mathbb{R} \to \{-1, 0, 1\}$ given by $f(x) = \lfloor x \rfloor - \lceil x \rceil$	○	●	○	○	○
$f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^3 + 1$	○	○	○	●	○

Solution:

1)

It is not injective because when rounded down, multiple x-values will give the same y-value. It is surjective since it continues on forever.

2)

$0, 0.5, 0.6,$ etc. are in $\mathbb{R}$, but rounded down to 0 are not in $\mathbb{Z}^+$ so this function isn't well defined.

3)

You have all odd numbers halved and rounded up make up the negative numbers, and same with even numbers. Since every integer times 2 is also an integer, every number in $\mathbb{Z}$ will have a corresponding x-value because of that. Also since integers times two only have one answer, you will not have multiple numbers in $\mathbb{N}$ give the same value

4)

This is not injective because almost all x-values will give the same y-value (-1). It is not surjective because a positive number rounded down minus same number rounded up will give −1. And same for negative numbers.

5)

No two natural numbers cubed plus one will give the same value, so this is injective. But there are a lot of natural numbers that is not hit.

Question 4

Consider the following three statements on the complete graph K_{2n}:

(a): K_{2n} has $\binom{2n}{2}$ edges

(b): K_{2n} has $2\binom{n}{2} + n^2$ edges

(c): K_{2n} has $n(2n-1)$ edges

Vælg en svarmulighed

- ○ (a) and (b) are true, but (c) is false
- ● (a) and (b) and (c) are all true
- ○ (a) and (c) are true, but (b) is false
- ○ (b) and (c) are true, but (a) is false
- ○ Precisely one of (a),(b),(c) is true

Solution:

(a) A complete graph is a graph where all vertices are connected, but only once. So the amount of edges can also be described as how many ways can you select two of $2n$ elements (how many ways can you select to vertices to connect in the graph)

(b) This can be verified using (a). Take $n = 10$ for example:

$$\begin{gathered}\binom{20}{2} = 190 \\ 2\binom{10}{2} + 100 = 2 \cdot 45 + 100 = 190\end{gathered} \tag{1}$$

This holds true, so (b) must be true as well.

(c) We can check this as well. For $n = 10$:

$$190 = 10 * 19 \tag{2}$$

This holds true as well.

Question 5

Consider a simple graph with degrees 2,2,3,3,3,3,3.

Vælg en svarmulighed

- ○ Such a graph exists, and any such graph has less than 19 edges
- ● Such a graph does not exist
- ○ Such a graph exists, and any such graph has precisely 19 edges
- ○ Such a graph exists, and any such graph has more than 19 edges
- ○ There exists such a graph with more than 19 edges and another with less than 19 edges

Solution:

Uneven amount of odd degrees, so it cannot exist.

Question 6

How many elements are in the union of four sets if each set has 200 elements, each pair of sets share 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets.

Vælg en svarmulighed

- ○ 395
- ○ 695
- ○ 495
- ● 595
- ○ None of these

Solution:

Imagine the sets A, B, C, D:

$$\begin{gathered} A+B+C+D-|A\cap B|-|A\cap C|-|A\cap D|-|B\cap C|-|B\cap D|-|C\cap D| \\ +\ |A\cap B\cap C|+|B\cap C\cap D|+|C\cap D\cap A|+|D\cap A\cap B| \\ -\ |A\cap B\cap C\cap D| = \\ 200+200+200+200-50-50-50-50-50-50 \\ +25+25+25+25-5=595 \end{gathered} \tag{3}$$

Question 7

Suppose that a, b, c, and m are positive integers such that $ac \equiv bc \mod m$. Which of the following must be true?

Vælg en svarmulighed

- ○ $a \equiv b \mod m$
- ● $a - b \in \{km : k \in \mathbb{Z}\}$ if $\gcd(c, m) = 1$
- ○ $c|m(a - b)$
- ○ $a \equiv b \mod cm$
- ○ None of these

Solution:

The solution means $a \equiv b \mod m$ but only if c, m are coprime. This is because $ac \equiv bc \mod m$ cant also be written as $m \mid c(a - b)$, so for our equation to hold, m must divide $a - b$ not c, so $\gcd(c, m) = 1$

Question 8

If $2a + 3 \equiv 2b + 3 \bmod m$, for positive integers a, b, and m, then which of the following does NOT necessarily have to be true?

Vælg en svarmulighed

- ○ $2a \equiv 2b \bmod m$
- ● $a = km + b$ for some $k \in \mathbb{Z}$
- ○ $14a \equiv 14b + 7m \bmod m$
- ○ All of these are true
- ○ $m|2(a-b)$

Solution:

Because for example: $a = 101, b = 100, m = 2$

$$\begin{gathered} 205 \equiv 203 \bmod 2 \\ 101 \neq 2k + 100 \text{ for some } k \in \mathbb{Z} \end{gathered} \tag{4}$$

Question 9

Which of these three are partitions of $\mathbb{Z} \times \mathbb{Z}$, that is, the set of ordered pairs of integers:

(a): the set of pairs (x, y) where x or y (or both) are odd; the set of pairs (x, y) where x and y are even;

(b): the set of pairs (x, y) where x and y are odd; the set of pairs (x, y) where x and y are even;

(c): the set of pairs (x, y) where x or y (or both) are odd; the set of pairs (x, y) where x or y (or both) are even.

Vælg en svarmulighed

○ None of them are

● (a) is a partition but (b),(c) are not

○ Precisely two of them are

○ (b) is a partition but (a),(c) are not

○ (c) is a partition but (a),(b) are not

Solution:

To be a partition, the following conditions must be met:
1. Every element $a \in S$ belongs to exactly one equivalence class $[a]_\sim$.
2. The equivalence classes are pairwise disjoint: if $[a]_\sim \neq [b]_\sim$, then $[a]_\sim \cap [b]_\sim = \emptyset$.
3. The union of all equivalence classes equals S.

So basically you can't have pairs appear more than once, and the union of all the pairs must be equal to, here, $\mathbb{Z}$.

a) This satisfies all three.
b) This satisfies the first two, but not the third.
c) This doesn't satisfy the first one.

Question 10

Consider the statement: There exist infinitely many simple graphs (that is, graphs with no loops and no multiple edges) such that all degrees are distinct. This statement can be proved to be

Vælg en svarmulighed

- ○ neither true nor false because some graphs have all degrees distinct and others do not
- ● false by the pigeonhole principle
- ○ false by giving a counterexample
- ○ true by the pigeonhole principle
- ○ true by induction

Solution:

If you have a graph of n vertices, and then add another one. It will need to have $n + 1$ edges, but for that to be true, it would have to connect to itself, which it can't (and for example the first edge, would need to connect to it as well).

In terms of Pigeonhole principle, we will end up having too many edges.

Question 11

Consider the polynomial $(2x^2 - 3y^3)^8$.

Vælg de rigtige svarmuligheder

	0	$2^4 3^4 \binom{8}{4}$	$-2^4 3^4 \binom{8}{4}$	$2^3 3^6 \binom{8}{4}$	$-2^4 3^6 \binom{8}{4}$
The coefficient of $x^8 y^{12}$ is	○	●	○	○	○
The coefficient of $x^6 y^9$ is	●	○	○	○	○

Solution:

You can use calculator to calculate this. But by intuition:

You know that $(x+y)^n$ will have coefficient: $\binom{n}{k} x^{n-k} y^k$

For $x^8 y^{12}$:

Here we have $n = 8, k = 8 - \frac{8}{2} = 4$. So coefficients are: $2^{n-k=4} 3^{k=4} \binom{n=8}{k=4}$. It will be positive as we multiple a negative number even number of times, so its positive.

For $x^6 y^9$. Here we have $n = 8, k = 8 - \frac{6}{2} = 5$ but $3 \cdot 5 \neq 9$ so this doesnt exist.

Question 12

The polynomial $x^n + 1$ is divisible by the polynomial $x + 1$ (where n is a positive integer)

Vælg en svarmulighed

- ○ for each $n \geq 1$
- ● for each odd $n \geq 1$, and for no even n
- ○ for each even $n \geq 4$, and for no odd n
- ○ for infinitely many, but not all, even n, and for no odd n
- ○ for $n = 1$ only

Solution:

Use calculator to solve this.

Question 13

The least number of cables required to connect ten computers to five printers to guarantee that, for every choice of five of the ten computers, these five computers can directly access five different printers is

Vælg en svarmulighed

- ○ 40
- ○ None of these
- ○ $\binom{10}{5}$
- ● 30
- ○ 50

Solution:

To guarantee that no matter the five computers we select they will connect to different printers, each printer must be connected to >5 computers so 6. And since there's 5 printers, its: $5 \cdot 6 = 30$

Question 14

Consider the following relations on $\{1, 2, 3, 4, 5, 6\}$:

R_1: $\{(1, 2), (2, 3), (1, 3), (4, 5), (5, 6), (4, 6)\}$.

R_2: $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 2), (3, 4), (5, 6), (1, 6)\}$.

R_3: $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (1, 3), (2, 4), (3, 5), (4, 6)\}$.

Recall that questions below may appear in random order.

Vælg de rigtige svarmuligheder

	an equivalence relation	a partial ordering	transitive and reflexive but not antisymmetric	transitive and antisymmetric but not reflexive	none of these
R_3 is	○	○	○	○	●
R_2 is	○	●	○	○	○
R_1 is	○	○	○	●	○

Solution:

Definitions:
1. **Reflexive:** $\forall a \in S, aRa$
2. **Symmetric:** $\forall a, b \in S$, if aRb then bRa
3. **Antisymmetric:** $\forall a, b \in S$, if aRb and bRa then $a = b$
4. **Transitive:** $\forall a, b, c \in S$, if aRb and bRc then aRc

A partial ordering is Reflexive, Antisymmetric and Transitive.

An equivalence relation is Reflexive, Symmetric and Transitive

$\boldsymbol{R_3}$ Not reflexive, not symmetric, is antisymmetric and not transitive (you have $(1, 2), (2, 4)$ but not $(1, 4)$)

$\boldsymbol{R_2}$ It is reflexive, antisymmetric and transitive. Thus, a partial ordering

$\boldsymbol{R_1}$ Not reflexive, not symmetric, is antisymmetric, is transitive.

Question 15

For each of the following values of n, determine the multiplicative inverse of $n \bmod 9$ or indicate that it does not exist.

Vælg de rigtige svarmuligheder

	Does not exist	0	1	2	3	4	5	6
$n = 6$	●	○	○	○	○	○	○	○
$n = 2$	○	○	○	○	○	○	●	○
$n = 7$	○	○	○	○	○	●	○	○

Solution:

The multiplicative inverse is the number for which you need to multiply n for $n \cdot x \equiv 1 \bmod 9$ here.

This can be calculated in Sympy.

$\boldsymbol{n = 6}$ No value

$\boldsymbol{n = 2}$ $2 \cdot 5 = 10 \equiv 1 \bmod 9$

$\boldsymbol{n = 7}$ $7 \cdot 4 = 28 \equiv 1 \bmod 9$

Question 16

Given a universal set U, which of the following sets are necessarily equal to $\left(\overline{B}-A\right) \cup \left(\overline{C}-A\right)$?

Vælg en svarmulighed

- ○ $\emptyset$
- ○ $\overline{(B \cup C)} - A$
- ○ $((U - B) \cup (U - C)) \cap A$
- ● $\overline{(B \cap C)} - A$
- ○ None of these

Solution:

The given equation mean: What is in either not B nor A, or not in C nor A.

This can be rewritten as what is not in both B and C and also not in A

Question 17

20 people are seated around a round table.

Vælg de rigtige svarmuligheder

	2^{20}	2^{19}	$20!$	$19!$	$19!/2$
Two seatings are considered identical if each person has the same two neighbors in the two seatings (but we don't care about left and right). The number of seatings is	○	○	○	○	●
Two seatings are considered identical if each person has the same left neighbor and the same right neighbor in the two seatings. The number of seatings is	○	○	○	●	○
Two seatings are considered identical if each person has the same left neighbor in the two seatings. The number of seatings is	○	○	○	●	○

Solution:

Can be calculated in Sympy!

But intuitively:

a) Here we have $\frac{20!}{20*2} = \frac{19!}{2}$. Because for every seating you have, you can rotate it 20 times and same neighbors, and you can mirror it, and same neighbors.

b) Same as before but only rotation now so: $\frac{20!}{20} = 19!$

c) This is the same as before as if you change a persons right neighbor, the now have a new left neighbor so you cant do that.

Question 18

Let a and b be positive integers. Which of the following integers can NOT necessarily be written as $as + bt$ for some integers s and t?

Vælg en svarmulighed

- ○ All of these can be written as $as + bt$ for some integers s and t
- ○ $\gcd(a,b)^2 - 17\ \mathrm{lcm}(a,b)$
- ● $\frac{\mathrm{lcm}(a,b)}{\gcd(a,b)}$
- ○ $\gcd(2a, 6b)$
- ○ $\gcd(a,b)$

Solution:

$\mathrm{lcm}(a,b)$ is the least common multiple of a, b so that means if you go any lower than that, it can't necessarily be divided by a, b so you cant necessarily write it as $as + bt$

Question 19

Which of the following gives a recursive definition of the number of ways to choose 3 elements from an n element set for $n \geq 3$?

Vælg en svarmulighed

- ○ $f(3) = 1$ and $f(n+1) = \binom{n}{2} f(n-1)$ for $n \geq 3$
- ○ None of these
- ○ $f(4) = 1$ and $f(n) = n^4 + f(n-1)$ for $n > 4$
- ○ $f(3) = 1$ and $f(n) = \binom{n}{3} + f(n-1)$ for $n > 3$
- ○ $f(n) = n^3$
- ● $f(3) = 1$ and $f(n) = \frac{(n-1)(n-2)}{2} + f(n-1)$ for $n > 3$

Question 20

For any integers n, m, where $2 \leq n \leq m$, the binomial coefficient $\binom{n+m}{m+2}$ equals

Vælg en svarmulighed

- ○ $\sum_{k=0}^{n} \binom{n}{k}\binom{m}{m-k}$
- ● $\sum_{k=2}^{n} \binom{n}{k}\binom{m}{m+2-k}$
- ○ $\sum_{k=0}^{n+2} \binom{n}{k}\binom{m}{n-k}$
- ○ None of these because we cannot use Vandermonde's identity in this case
- ○ $\sum_{k=1}^{n} \binom{n}{k}\binom{m}{m+1-k}$

Solution:

Because we know: $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$

Question 21

The number of derangements of 1,2,3,4,5,6,7 ending with 1,2,3 in some order is

Vælg en svarmulighed

- ○ $3!(4! - 1)$
- ○ $3!4!/2$
- ○ $3!4!$
- ● $3!(4! - 3!)$
- ○ None of these

Solution:

Can be calculated in Sympy.

Intuitively:

There are $3!$ ways to order $1, 2, 3$. And if it must end in that, there are 4 more numbers to order, which have $4!$ numbers of ways to be ordered. Since it is a derangement though, this means that no number can have its original position, so we must count out the positions where 4 is in the 4th spot, so $-3!$. So it becomes $3!(4! - 3!)$

Question 22

Consider the following system of congruences:

$x \equiv 1 \bmod 2$

$x \equiv 4 \bmod 5$

$x \equiv 3 \bmod 7$

Indicate the set of all solutions to the above system of congruences.

Vælg en svarmulighed

- ○ $\{12 + 70k \mid k \in \mathbb{Z}\}$
- ○ $\{8 + 70k \mid k \in \mathbb{Z}\}$
- ○ $\{1 + 2k \mid k \in \mathbb{Z}\} \cup \{4 + 5k \mid k \in \mathbb{Z}\} \cup \{3 + 7k \mid k \in \mathbb{Z}\}$
- ○ $\{70 + 12k \mid k \in \mathbb{Z}\}$
- ○ $\{8 + 14k \mid k \in \mathbb{Z}\}$
- ● $\{59 + 70k \mid k \in \mathbb{Z}\}$
- ○ None of these

Solution:

Can be solved using Sympy.

Also calculate for each of the formulas whether 2 divides it minus 1, 5 divides it minus 4 and 7 divides it minus 3.

Question 23

It is possible to prove that a simple graph on $n \geq 1$ vertices has at most $\binom{n}{2}$ edges using mathematical induction. Here we take $\binom{1}{2}$ to be equal to zero.

By choosing some of the following text fragments and putting them in the correct order, a proof by induction for the above statement can be created.

A) To prove the inductive step, assume that every simple graph on n vertices has at most $\binom{n}{2}$ edges, for every integer $n \geq 1$. We will show that this implies that every simple graph on $n+1$ vertices has at most $\binom{n+1}{2}$ edges.

B) To prove the inductive step, assume that every simple graph on n vertices has at most $\binom{n}{2}$ edges, for some fixed integer $n \geq 1$. We will show that this implies that every simple graph on $n+1$ vertices has at most $\binom{n+1}{2}$ edges.

C) The statement now follows from the principle of mathematical induction.

D) Since v had at most n edges incident to it, we have that G has at most $n + \binom{n}{2}$ edges. Moreover, $n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{1}{2}(2n + n^2 - n) = \frac{1}{2}(n^2 + n) = \frac{(n+1)n}{2} = \binom{n+1}{2}$.

E) We prove the statement by induction. The base case is $n = 1$. A simple graph on 1 vertex cannot have any edges, so it has at most $\binom{1}{2} = 0$ edges.

F) Let G be a simple graph on $n+1$ vertices. Pick any vertex $v \in V(G)$ and note that v has at most n edges incident to it since there are n vertices it could be adjacent to. Let G' be the graph obtained from G by removing v and all the edges incident to v. Then G' is a simple graph on n vertices and thus it has at most $\binom{n}{2}$ edges by our induction hypothesis.

G) Let G be a simple graph on n vertices. Construct a simple graph G' on $n+1$ vertices by adding a new vertex v to G and making it adjacent to an arbitrary subset S of vertices of G. Since G has n vertices, we have that $|S| \leq n$ and so v is incident to at most n edges. Therefore G' has at most $n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \frac{1}{2}(2n + n^2 - n) = \frac{1}{2}(n^2 + n) = \frac{(n+1)n}{2} = \binom{n+1}{2}$ edges.

PLEASE NOTE THAT THE QUESTIONS AND ANSWERS BELOW APPEAR IN RANDOM ORDER. PLEASE READ THEM CAREFULLY.

Vælg de rigtige svarmuligheder

	A	B	C	D	E	F	G	No more fragments
The first text fragment of the proof is	○	○	○	○	●	○	○	○
The second text fragment of the proof is	○	●	○	○	○	○	○	○
The third text fragment of the proof is	○	○	○	○	○	●	○	○
The fourth text fragment of the proof is	○	○	○	●	○	○	○	○
The fifth text fragment of the proof is	○	○	●	○	○	○	○	○

Solution:

First is **E)** because it is our base step.

Then it is **B)** for the start of the inductive step. It is not **A)** because we calculate induction for some *fixed* integer, not every integer.

Then is **F)** because it makes more sense than **G)**

Then is **D)** because calculations

Then is **C)** because there is another option left and this also concludes stuff.

Question 24

Let $P(x)$ denote the statement "x is prime". Which of the following statements can be written in predicate logic as the following:

$\forall x \in \mathbb{Z}(x > 1 \rightarrow (\exists y \in \mathbb{N}(P(y) \wedge x < y < 2x)))$

Vælg en svarmulighed

- ○ For every integer x greater than 1, every number strictly between x and $2x$ is prime.
- ○ None of these
- ○ For every integer x, either $x > 1$ or there is a prime y such that $x < y < 2x$.
- ○ For any integer $x > 1$ and any prime y, if $x > 1$ then $x < y < 2x$.
- ● For every integer $x > 1$, there is a prime y such that $x < y < 2x$.
- ○ There exist infinitely many primes.

Solution:

This is just using words to describe the statement

Question 25

Let a and b be two positive integers with $ab = 5292 = 2^2 3^3 7^2$. Which of the following values CANNOT be the greatest common divisor of a and b?

Vælg en svarmulighed

- ○ 1
- ○ 3
- ● 36
- ○ 42
- ○ All of these are possible values for $\gcd(a, b)$

Solution:

36 can't be gcd because, even though you can construct it using the primes, you do: $2 \cdot 2 \cdot 3 \cdot 3$, and then you wouldn't be able to construct it again, meaning it would only be able to divide one of a, b

Question 26

Consider the number of ways we can put $n + 1$ distinguishable balls (meaning that we can tell the difference between them) into n distinct boxes.

Vælg de rigtige svarmuligheder

	none of these	$2n!$	$\binom{n+1}{2}n!$	$n^{n+1} - 2n!$	$n^{n+1} - \binom{n+1}{2}n!$
If at least one box is empty, the number of ways is	○	○	○	○	●
If no box is empty, the number of ways is	○	○	●	○	○

Solution:

For the non-empty one first:

You first chose 2 of the $n + 1$ balls, these will go into the same container. Then you put the balls into the container (the two balls just acting as one)

For the non-empty one:

You can arrange the balls in n ways in the boxes, but then you must subtract all the ways to order the balls when there are no empty boxes as there must be at least one empty box.