01017 E23 Exam Discrete Mathematics

Der anvendes en scoringsalgoritme, som er baseret på "One best answer"

Dette betyder følgende:

Der er altid netop ét svar som er korrekt
Studerende kan kun vælge ét svar per spørgsmål
Hvert rigtigt svar giver 1 point
Hvert forkert svar giver 0 point (der benyttes IKKE negative point)

The following approach to scoring responses is implemented and is based on "One best answer"

There is always only one correct answer
Students are only able to select one answer per question
Every correct answer corresponds to 1 point
Every incorrect answer corresponds to 0 points (incorrect answers do not result in subtraction of points)

Define $f(n) = \sum_{k=0}^{n} k^2$.
It is possible to prove by induction that $f(n) = \frac{n(n+1)(2n+1)}{6}$ holds for all nonnegative integers n.

By choosing 4 of the following 8 text fragments and putting them in the correct order, a proof by induction for the above statement can be created.

A) To prove the induction step, we assume that $f(n) = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$. We now prove that $f(n+1) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$ holds for all $n \in \mathbb{N}$.

B) We prove the statement by induction. The base case is $n = 0$. For $n = 0$, we see that $f(n) = \sum_{k=0}^{0} k^2 = 0$ and also that $\frac{n(n+1)(2n+1)}{6} = \frac{0 \cdot (0+1)(2 \cdot 0+1)}{6} = 0$. This proves the base case.

C) We have that

$$\begin{aligned} f(n+1) &= \sum_{k=0}^{n+1} k^2 \\ &= f(n) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \text{ by the induction hypothesis} \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\ &= \frac{(n+1)[2n^2 + 7n + 6]}{6} \\ &= \frac{(n+1)[(n+2)(2n+3)]}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} \end{aligned}$$

which is what we wanted to prove. This concludes the induction step.

D) We have that

$$\begin{aligned} f(n) &= \sum_{k=0}^{n} k^2 \\ &= f(n+1) - (n+1)^2 \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} - (n+1)^2 \text{ by the induction hypothesis} \\ &= \frac{(n+1)(n+2)(2n+3) - 6(n+1)^2}{6} \\ &= \frac{(n+1)[(n+2)(2n+3) - 6(n+1)]}{6} \\ &= \frac{(n+1)[2n^2 + n]}{6} \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

which is what we wanted to prove. This concludes the induction step.

E) To prove the induction step, we assume that $f(n) = \frac{n(n+1)(2n+1)}{6}$ holds for some $n \in \mathbb{N}$. We will now prove that under this assumption, $f(n+1) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$.

F) We prove the statement by induction. The base case is $n = 1$. For $n = 1$, we see that $f(n) = \sum_{k=0}^{1} k^2 = 1$ and also that $\frac{n(n+1)(2n+1)}{6} = \frac{1\cdot(1+1)(2\cdot1+1)}{6} = \frac{1\cdot2\cdot3}{6} = 1$. This proves the base case.

G) The statement now follows from the principle of mathematical induction.

H) To prove the induction step, we assume that $f(n+1) = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$ holds for some $n \in \mathbb{N}$. We will now prove that under this assumption, $f(n) = \frac{n(n+1)(2n+1)}{6}$.

Select the correct answers	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**
The first text fragment of the proof is part	○	○	○	○	○	○	○	○
The second fragment is	○	○	○	○	○	○	○	○
The third fragment is	○	○	○	○	○	○	○	○
The fourth fragment is	○	○	○	○	○	○	○	○

For each of the following values of n, determine the multiplicative inverse of $n \pmod 6$ or indicate that it does not exist.

Select the correct answers

	Does not exist	0	1	2	3	4	5
$n = 0$	○	○	○	○	○	○	○
$n = 2$	○	○	○	○	○	○	○
$n = 5$	○	○	○	○	○	○	○

Define the function $f(n)$ for integers $n \geq 3$ recursively as $f(3) = 1$ and $f(n) = \frac{(n-1)(n-2)}{2} + f(n-1)$ for $n > 3$. Which of the following functions defined on integers greater than or equal to 3 are equal to f?

Choose one answer

- $\binom{n}{2}$
- $|A|$ where $A = \{(k_1, k_2) | k_1, k_2 \in \mathbb{Z}^+ \wedge k_1 < k_2 \leq n\}$
- None of these
- $\frac{1}{6}\left(n^3 - 6n^2 + 10n - 6\right)$
- $\frac{1}{420}\left(39n^7 - 1687n^6 + 30555n^5 - 300265n^4 + 1728076n^3 - 5819338n^2 + 10607180n - 8064000\right)$
- The number of ways to choose 3 things from a set of size n where the order does not matter
- $\frac{n^3 - n}{6}$

Consider the following system of congruences:

$x \equiv 2 \bmod 3$

$x \equiv 3 \bmod 5$

$x \equiv 5 \bmod 7$

Indicate the set of all solutions to the above system of congruences.

Choose one answer

- $\{105 + 30k \mid k \in \mathbb{Z}\}$
- $\{30 + 105k \mid k \in \mathbb{Z}\}$
- $\{2 + 3k \mid k \in \mathbb{Z}\} \cup \{3 + 5k \mid k \in \mathbb{Z}\} \cup \{5 + 7k \mid k \in \mathbb{Z}\}$
- $\{30 + 7k \mid k \in \mathbb{Z}\}$
- None of these
- $\{10 + 105k \mid k \in \mathbb{Z}\}$
- $\{68 + 105k \mid k \in \mathbb{Z}\}$

Consider the following statement:

"Given any two distinct real numbers, there is a real number which sits strictly between them."

Which of the following formulas written in predicate logic is a correct formalization of the above statement?

Choose one answer

- ○ None of these
- ○ $\forall x \in \mathbb{R}\ \forall z \in \mathbb{R}\ \exists y \in \mathbb{R}(x < y < z \vee z < y < z)$
- ○ $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}\ \forall z \in \mathbb{R}(x \neq z \rightarrow (x < y < z \vee z < y < x))$
- ○ $\exists x \in \mathbb{R}\ \exists z \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x < y < z \vee z < y < x)$
- ○ $\forall x \in \mathbb{R}\ \forall z \in \mathbb{R}\ (x \neq z \rightarrow (\exists y \in \mathbb{R}\ (x < y < z) \vee \exists y \in \mathbb{R}\ (z < y < x)))$

For each of the following, determine whether it is surjective/injective, or not a well defined function.

Select the correct answers

	Not a well defined function	Well defined but neither surjective nor injective	Surjective but not injective	Injective but not surjective	Both surjective and injective
$f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$	○	○	○	○	○
$f : \mathbb{Z} \to \mathbb{N}$ given by $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x - 1 & \text{if } x < 0 \end{cases}$	○	○	○	○	○
$f : \mathbb{Z} \to \mathbb{N}$ given by $f(x) = \begin{cases} 2x - 1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$	○	○	○	○	○
$f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = \lfloor \frac{x}{2} \rfloor$	○	○	○	○	○
$f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x!$	○	○	○	○	○

Given a universal set U, which of the following sets are equal to $\overline{A \cap (B \cup C)}$?

Choose one answer

- None of these
- $(A \cap (B \cup C)) - U$
- $\overline{A} \cap \left(\overline{B} \cup \overline{C}\right)$
- $A \cup (B \cap C)$
- $(U - (A \cap B)) \cap (U - (A \cap C))$

If $a, b,$ and c are positive integers such that $a|bc$, then which of the following must be true?

Choose one answer

- ○ $\gcd(b, c)|a$
- ○ None of these
- ○ If b is prime, then $a|b$ or $a|c$.
- ○ $a|\mathrm{lcm}(b, c)$
- ○ If $\gcd(a, b) = 1$ then $a|c$.
- ○ $a|b$ or $a|c$

If $a \equiv b \bmod m$, for positive integers $a, b,$ and m, then which of the following must be true?

Choose one answer

- None of these
- $a - b = m$
- There exist $s, t \in \mathbb{Z}$ such that $sa + tb = m$.
- $m|(a + b)$
- $a = km + b$ for some $k \in \mathbb{Z}$

Let a and b be positive integers and let $d = \gcd(a, b)$ and $m = \operatorname{lcm}(a, b)$. Then which of the following are true?

Choose one answer

- $\operatorname{lcm}(m, d) = ab$
- $ab = md$
- $ab|m$
- None of these
- $\gcd(d, m) = 1$
- There exist $s, t \in \mathbb{Z}$ such that $sa + tb = 1$.

Let n be a natural number. Then which of the following are true?

Choose one answer

- ○ $n^2 > n$ for infinitely many n
- ○ None of these
- ○ $n^3 > n$ for all n
- ○ $n^2 > n$ for all n

Consider all permutations of abcde.

Select the correct answers

	36	42	64	81	114	None of these
How many do not contain abc?	○	○	○	○	○	○
How many contain ab or bc?	○	○	○	○	○	○
How many contain ab or bc but not abc?	○	○	○	○	○	○

The coefficient of $x^7 y^2$ in $(2x + 3y)^9$ is

Choose one answer

- ○ $2^9 3^4$
- ○ $2^9 3^9$
- ○ None of these
- ○ $2^{10} 3^{10}$
- ○ $2^8 3^{10}$
- ○ $2^8 3^9$

The number D_6 of derangements of 6 elements is

Choose one answer

- 250
- None of these
- 300
- 265
- $6! - 5! + 4! - 3! + 2! - 1!$
- 280

The number of edges in the 4-cube graph is

Choose one answer

- 32
- 25
- None of these
- 16
- 20
- 30

$n2^{n-1}$ equals

Choose one answer

- ○ $\sum_{k=1}^{n} \binom{n}{k} k$
- ○ $\sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1}$
- ○ $\sum_{k=1}^{n} 2^k$
- ○ $\sum_{k=0}^{n-1} 2^k$
- ○ None of these

Consider the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$. How many numbers must be selected to guarantee that some of them add up to 16?

Choose one answer

- None of these
- 6
- 4
- 5
- 7

For each relation on the set of four distinct elements a, b, c, d below decide which property it has.

Select the correct answers

	reflexive	symmetric	antisymmetric	transitive	none of these
$\{(a,b),(b,a),(b,c),(c,b),(c,d),(d,a),(a,d)\}$	○	○	○	○	○
$\{(a,a),(b,b),(c,c),(d,d),(c,d),(d,c),(b,c)\}$	○	○	○	○	○
$\{(a,b),(b,c),(c,d),(a,c)\}$	○	○	○	○	○

Let A be the set $\{1, 2, \ldots, 80\}$.

Select the correct answers

	2^{60}	$\binom{60}{20}$	$\binom{80}{20}$	$\binom{80}{40}$	2^{79}	None of these
How many subsets of A have size precisely 20.	○	○	○	○	○	○
How many subsets of A have no element greater than 60	○	○	○	○	○	○
How many subsets of A have an even number of elements.	○	○	○	○	○	○

Suppose 6 people A,B,C,D,E,F are seated around a round table (where rotating the entire seating arrangement counts as a different arrangement).

Select the correct answers

	$6! - 12 \cdot 4!$	$6! - 5!$	$6! - 6 \cdot 4!$	$6!/5$	None of these
In how many ways can they be seated such that A and B are not seated opposite to each other?	○	○	○	○	○
In how many ways can they be seated such that A and B are not seated next to each other?	○	○	○	○	○

Euclidean Algorithm for Polynomials

Below are the results of running the Euclidean algorithm on the polynomials $N(x)$ and $M(x)$. Here, $R_k(x)$ denotes the remainder found in the k^{th} step of the algorithm.

k	$R_k(x)$
1	$-2x^2-6x+8$
2	$62x-62$
3	0

What is the greatest common divisor of $N(x)$ and $M(x)$?

Choose one answer

- None of these
- $-2x^2 - 6x + 8$
- $62x - 62$
- 0

How many common roots do $N(x)$ and $M(x)$ have?

Choose one answer

- ○ 2
- ○ 0
- ○ 1
- ○ None of these