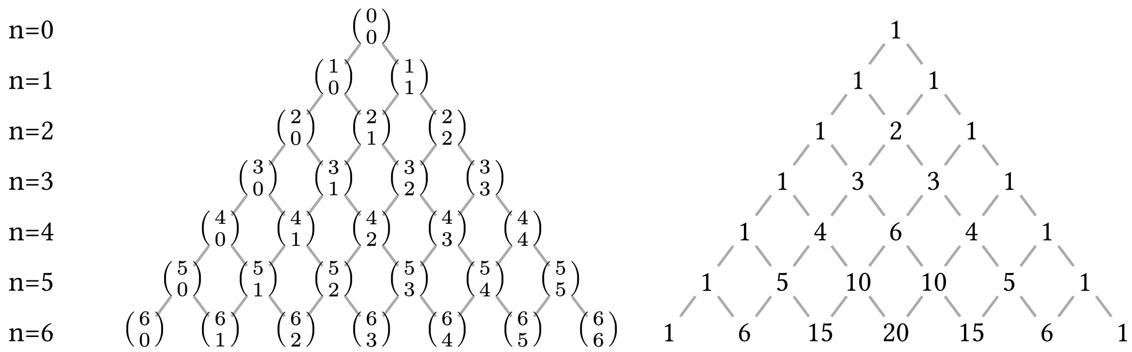
Binomial coefficients, formula and identities

Binomial coefficient

From last week:

$C(n, k) =$ the number of k-combinations from an n-set. Or the number of ways to select k elements from an n-set. Or the number of k-subsets of an n-set. Formula: $\frac{n!}{k!-(n-k)!} = \binom{n}{k}$ for $0 \leq k \leq n$

Pascal's triangle



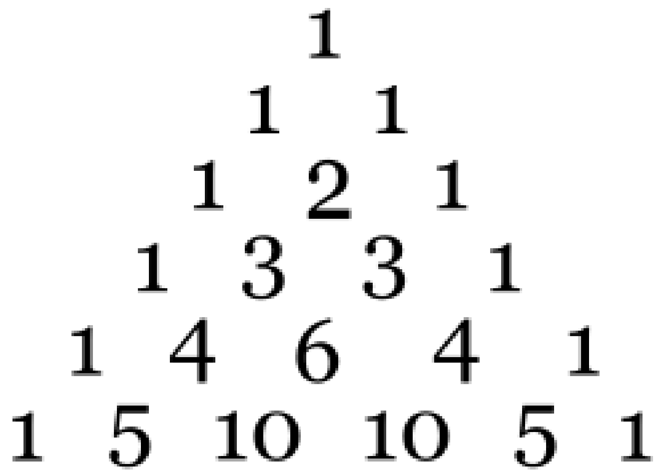


Figure 1: Another form of Pascals triangle. It is mirrored

Binomial identity:

$$\binom{n}{k} = \binom{n}{n-k}$$

You can do analytic proof (mathematic proof using the formulas) or the combinatorial proof (count it in one way to get first result, then in a different way to get the other result).

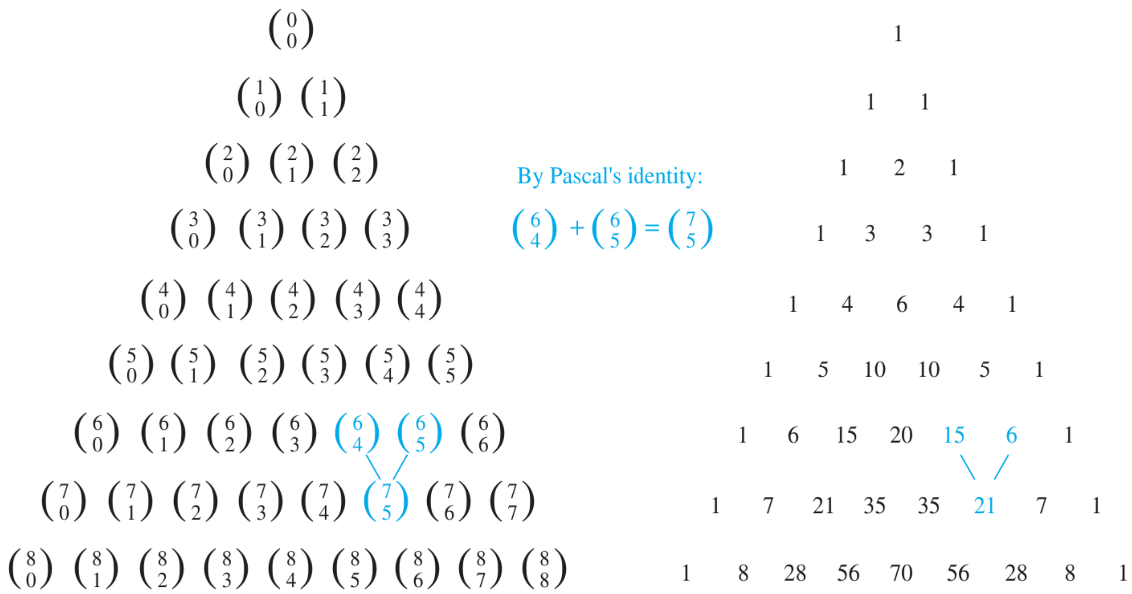
Combinatorial proof for $\binom{n}{k} = \binom{n}{n-k}$: You could say that for n people, you ask k people to walk out the door, or you could ask `n-k` people to stay in the room.

In Pascal's triangle, adding the rows:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + (n; n) = \sum_{k=0}^{n} \binom{n}{k} = 2^k$$

Binomial identity

For $\binom{n}{k}$ you can also write $\binom{n-1}{k} + \binom{n-1}{k-1}$. It is called Pascal's identity.



Binomial formula

$$(x+y)^1 = x^1 + y^1$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^3 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Note for each $x^a y^b$, you have $(x+y)^{a+b}$

You also have that the coefficients (for $(x+y)^4$ you have 1,4,6,4,1) they follow Pascal's triangle

The formula

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\end{aligned}$$

$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$

Proof by induction with respect to `n`

Basis step: Put $n = 1$ and prove that the formular is true:

$$\begin{aligned}(x+y)^1 &= \binom{1}{0} \cdot x^0 \cdot y^1 + \binom{1}{1} \cdot x^1 \cdot y^0 \\ &= y + x\end{aligned}$$

Induction step: Assume true for `n-1`, then prove that it's then true for `n`:

$$(x+y)^{n-1} = \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k}$$

$$
\begin{aligned}
(x+y)^n &= (x+y)\cdot(x+y)^{n-1} \\
&= (x+y)\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k} \\
&= x^n + \sum_{k=0}^{n-2}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=1}^{n-1}\binom{n-1}{k}x^k y^{n-k} + y^n \\
&\text{Replace k+1 with k} \\
&= x^n + \sum_{k=1}^{n-1}\binom{n-1}{k-1}x^k y^{n-k} + \sum_{k=1}^{n-1}\binom{n-1}{k}x^k y^{n-k} + y^n \\
&\text{Note from pascals identity: } \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\end{aligned}
$$

Combinatorial Proof:

Consider expanding $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ (n factors).

To get a term $x^{n-k}y^k$, we must:

- Choose k of the n factors to contribute a y (the rest contribute x)
- There are $\binom{n}{k}$ ways to do this

Therefore, the coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$.

Van der Mande

$\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$